CSE 483: Mobile Robotics

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Incremental QR Factorization by Givens Rotation

We have reduced our SAM problem to a least squared fitting problem after linearizing our non linear system as follows:

$$\|A\delta - b\|^2 \tag{1}$$

where A is the observation Jacobian of dimention m x n. δ is variable vector (n x 1) which we want to find and b is a constant vector. (m x 1)

$$A\delta = b \tag{2}$$

In a non-incremental way we would have found the pseudo inverse of A to get δ. Finding Inverse of large matrix A is computationally inefficient. So we use QR decomposition method to solve for δ which can easily be incrementalized.

1 QR Decomposition

Since A is real matrix. QR decomposition of A is always possible.

$$A = QR \tag{3}$$

where columns of Q are orthonormal vectors. R is upper triangular matrix.

$$Q^T A = Q^T QR$$

Since, Q is orthonormal,

$$QQ^T = I$$

$$Q^T A = R \tag{4}$$

Now,

$$\begin{aligned}
\|A\delta - b\|^2 &= (A\delta - b)^T (A\delta - b) \\
&= (A\delta - b)^T (QQ^T)(A\delta - b) \\
&= ((A\delta - b)^T Q)(Q^T (A\delta - b)) \\
&= ((A\delta - b)^T Q)(Q^T (A\delta - b)) \\
&= \left\|Q^T (A\delta - b)\right\|^2 \\
&= \left\|R\delta - Q^T b\right\|^2 \\
&= \|R\delta - c\|^2
\end{aligned}$$

Let $\mathbf{Q^T b}$ = c where c (m x 1) can be decomposed in d and e where d has first n elements of c and e has the remaining elements. R (m x n) can be decomposed into upper triangular square matrix R (n x n) (new R) and a null matrix (0) as follows:

$$\|A\delta - b\|^2 = \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} \delta - \begin{bmatrix} d \\ e \end{bmatrix} \right\|^2 \tag{5}$$

Now, we need to find Q which would make R an upper triangular matrix. This can be achieved by making all the elements below diagonal zero one by one using Givens Rotation. Each $\mathbf{Q_i}$ would make one element in A zero. The process starts from the left-most non-zero entry, and proceeds column wise bottom up.
Let p be the total number of operations.

$$R = (Q_1 Q_2 Q_3 .... Q_p)^T A = Q^T A$$

Once we have the the R which is upper triangular, solving for δ is very fast. As the n^{th} rows would give a linear equation with one variable. $(n-1)^{th}$ row would give linear equation with 2 variable out of which 1 is already solved in the previous equation. So we can solve for other one. $(n-2)^{th}$ row would give 3 variable linear equation and out of which 2 have been already solved. And so on...

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta l_1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ e_1 \\ e_2 \end{bmatrix}$$

$$\begin{aligned} f * \delta l_1 &= d_3 \\ d * \delta x_2 + e * \delta l_1 &= d_2 \\ a * \delta x_1 + b * \delta x_2 + c * \delta l_1 &= d_1 \end{aligned}$$

Now task is to get R in incremental form.

2 Givens Rotation

$$Let, A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

To make the element $\mathbf{a_{i,k}}$ zero we need to consider elements $\mathbf{a_{i,k}}, \mathbf{a_{i-1,k}}$ to find the rotation matrix. In the orthonormal matrix we need to set $\mathbf{q_{i-1,i-1}}, \mathbf{q_{i-1,i}}, \mathbf{q_{i,i-1}}, \mathbf{q_{i,i}}$ with the rotation matrix.

eg We want $\mathbf{a_{31}}$ to become 0. $\mathbf{a_{3,1} = g}$ $\mathbf{a_{2,1} = d}$ would be considered.

$$Q_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{6}$$

$$where, \sin\phi = \frac{g}{\sqrt{d^2+g^2}} \tag{7}$$

$$\cos\phi = \frac{-d}{\sqrt{d^2+g^2}} \tag{8}$$

$$\begin{aligned} Q_1^T A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} a & b' & c' \\ d\cos\phi - g\sin\phi & e' & f' \\ d\sin\phi + g\cos\phi & h' & i' \end{bmatrix} \\ &= \begin{bmatrix} a & b' & c' \\ d' & e' & f' \\ 0 & h' & i' \end{bmatrix} \end{aligned}$$

Similarly, we can apply further transformations $\mathbf{Q_2}, \mathbf{Q_3}, ...$ on A to finally get an upper triangular matrix

$$R = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$$

3 Incremental Factorization Updates

In the first iteration we would make all the elements below diagonal zero. But when a new measurement arrives, the previous factorization can be directly modified by QR updating instead of updating and refactoring entire A.

Let $\mathbf{w^T}$ be the new measurement row.
Let γ be the RHS factor to be appended to d.
The new system is given by (not yet correctly factorized):

$$\begin{bmatrix} Q^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ w^T \end{bmatrix} = \begin{bmatrix} R \\ w^T \end{bmatrix} \tag{9}$$

$$New\ RHS: \begin{bmatrix} D \\ \gamma \end{bmatrix}$$

Now Givens rotation can be used to zero out the new added row and thus find the updated factor R'. Same rotation would be applied to RHS as well.

NOTE:

- New variables i.e. new landmark or new state are added to the QR factorization by expanding the factor R by the appropriate number of empty columns and rows. (n would increase) This expansion is simply done before new measurement rows containing the new variables are added. At the same time, the RHS d is augmented by the same number of zeros.
- Whereas adding a new observation adds a new row to A(m would increase).
- All the Jacobian in the A are evaluated/linearized at a some estimate of state and landmark. At the end of each cycle we would get better estimate for the same. But since the change wouldn't be significant and would require very high computation to recreate A, we simply use incremental and doesn't change the Jacobians. But after large number of steps (say 100), it is preferred to recompute entire A again make and use Given Rotations to make it upper triangular followed by same incremental method till next 100 iteration.